Inertial circles - visualizing the Coriolis force: GFD VI

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Abstract

We study the trajectories of dry ice pucks launched over the surface of a smooth, rotating parabola viewed from both inertial and rotating frames. Experiments are described which are designed to help us come to a deeper understanding of frames of reference and the Coriolis force.

1 Introduction

In this laboratory we study the trajectory of a 'frictionless' dry ice puck sliding over a smooth parabolic surface. The parabola available in the lab was manufactured by pouring resin in to a mold on a table turning at a rate $f = 2\Omega = 3\ rad\ s^{-1}$ and allowing it to solidify forming a highly smooth surface: it is a metre in diameter and a centimeter or so deeper in the center than at the periphery - see Fig.1 and section 4.2 of the appendix.[1]

Place the parabola on the rotating table and, for the moment, do not spin the table up. Launch the puck along a radius toward its center. The puck oscillates along a straight line passing through the origin. Its trajectory is governed by the equation:

$$\frac{d^2r}{dt^2} = -g\frac{dh}{dr} \tag{1}$$

where r is the distance of the puck from the center of the parabola, g is the acceleration due to gravity and $h(r)$ is the shape of the parabolic surface. The restoring force on the puck is just gravity resolved in the direction of the surface. Because the surface is parabolic i.e. of the form $h = h(0) + ar^2$, where a is a constant and $h(0)$ is the depth of the parabola at the centre, then $\frac{dh}{dr} = 2ar$ — thus the restoring force in Eq.(1) is linear in r increasing toward the edge of the parabola where the surface tilt is most pronounced. Because of this linearity in the restoring force, the puck performs *simple harmonic motion*.

[1] The procedure used to manufacture the parabola is described here: http://paoc.mit.edu/labweb/parabolic_surface.htm.

Figure 1: Experiments with ball bearings and dry ice 'pucks' on a rotating parabola. A corotating camera views the scene from above.

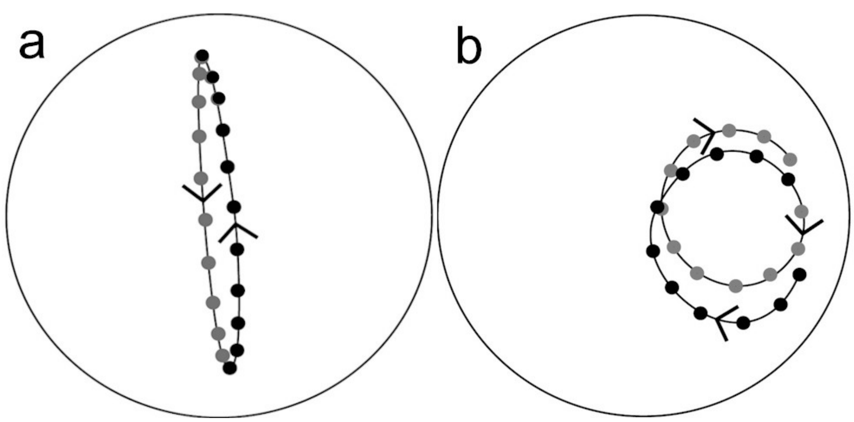

Figure 2: Trajectory of the puck on the rotating parabolic surface in (a) the inertial frame and (b) the rotating frame of reference. The parabola is rotating in an anticlockwise (cyclonic) sense.

Now spin up the parabolic surface by rotating the table at rate $f = 2\Omega = 3\ rad\ s^{-1}$ (the speed used to manufacture the parabola), in an anticlockwise direction. If the surface of the parabola is indeed frictionless, then the puck, launched as before, will perform simple harmonic motion in a straight line even though the parabola is rotating beneath. This can be seen in Fig.2a where the trajectory of an observed puck in the inertial (fixed) frame of reference is plotted. Note that because it is impossible to reduce frictional effects to zero, in practice the straight line is dragged out in to an ellipse.

But what do we observe if we place ourselves in a frame of reference rotating with the table? The trajectory of the puck when viewed in the rotating frame (recorded by an overhead camera co-rotating with the parabola) is shown in Fig.2b. The puck moves in circles! The equation that governs the trajectory in the rotating frame is very different from Eq.(1) and involves, as we shall see, the Coriolis force which 'deflects the puck to the right'. The circular trajectories — which are called 'inertial circles' — are commonly observed in the atmosphere and ocean. They are a consequence of observing the motion in a rotating frame of reference.

The parabolic surface used in our experiments has the shape that the free surface of a fluid takes up in solid body rotation in a tank rotating at rate Ω - see section 4.2:

$$h = h(0) + \frac{\Omega^2 r^2}{2g} \tag{2}$$

where Ω is the rotation rate of the table. The surface defined by Eq.(2) is an equipotential surface and so a body carefully placed on it at rest should remain at rest. Indeed if we place a ball-bearing on the parabolic surface rotating at speed Ω, then we see that it does not fall in to the center but instead finds a state of rest in which the component of gravitational acceleration acting on it resolved along the parabolic surface, g_H, is exactly balanced by the outward-directed horizontal component of the centrifugal acceleration resolved in the surface, $(\Omega^2 r)_H$, as sketched in Fig. 3.

In the case that h is given by Eq.(2), the restoring force $-g\frac{dh}{dr} = -\Omega^2 r$, Eq.(1) takes on the form:

$$\frac{d^2 r}{dt^2} = -\Omega^2 r. \tag{3}$$

and describes simple harmonic motion with frequency Ω.

The experiments we now describe are designed to help us come to a deeper understanding of frames of reference and the Coriolis acceleration. We use our rotating parabolic surface in conjunction with ball-bearings and a frictionless dry ice 'puck' to study trajectories in the inertial and rotating frames.

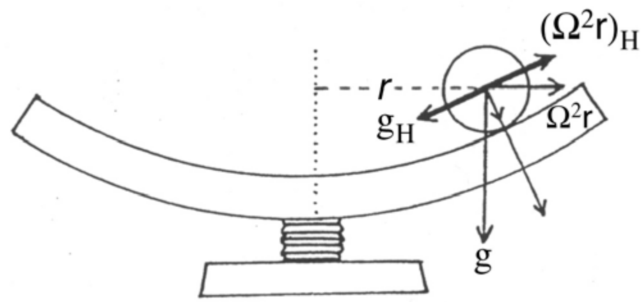

Figure 3: If a parabola of the form given by Eq.(2) is spun at rate Ω, then a ball carefully placed on it at rest does not fall in to the center but remains at rest.

2 Experimental procedure

We can now play games with the dry ice puck and study its trajectory on the parabolic turntable, both in the rotating and laboratory frames. It is useful to view the puck from the rotating frame using an overhead co-rotating camera. The following are useful reference experiments:

1. set up the parabola on the rotating table and adjust the speed of rotation to match that which was used to manufacture it. The exact Ω can be checked by placing a ball bearing on the parabola so that is motionless in the rotating frame of reference: at the 'correct' Ω the ball can be made motionless — at which point the balance of forces as sketched Fig.(3) — without it riding up or down the surface. In the laboratory frame the ball follows a circular orbit around the center of the dish.

2. launch the puck on a trajectory that lies within a fixed vertical plane containing the axis of rotation of the parabolic dish. Viewed from the laboratory the puck moves backwards and forwards along a straight line (the straight line will expand out in to an ellipse if the frictional coupling between the puck and the rotating disc is not negligible - see Fig.2). When viewed in the rotating frame, however, the trajectory appears as a circle tangent to the straight line. This is the experiment from which the results presented in Fig.2 are shown. These circles are called 'inertial circles' - see theory below.

   Compute the period of the oscillations of the puck in the inertial and rotating frames. How do they compare to one-another and Ω?

   Compute the trajectory of the puck by using the theory of inertial circles presented in section 3 and compare to the observed trajectory - see 4. below

3. again place the puck so that it appears stationary in the rotating frame, and then slightly perturb it. In the rotating frame the puck undergoes inertial oscillations consisting of small circular orbits passing through the initial position of the unperturbed puck.

4. use the particle tracking software to compute the trajectories of the particles and compare them to the theory of inertial circles presented below.

3 Theory of Inertial circles

It is straightforward to analyze the motion of the puck in our experiment. We adopt a Cartesian (x, y) coordinate in the rotating frame of reference whose origin is at the center of the parabolic surface. The velocity of the puck on the surface is $\mathbf{u}_{rot} = (u, v)$ where $u_{rot} = dx/dt$ and $v_{rot} = dy/dt$. Further we assume that z increases upwards in the direction of Ω.

3.1 Rotating frame

The law of motion of the puck traversing the frictionless parabolic surface are given by Eq.(15) of the appendix, which we write out again here:

$$\frac{d\mathbf{u}_{rot}}{dt} = -2\mathbf{\Omega} \times \mathbf{u}_{rot}$$

Let's write out Eq.(15) in component form. Noting that:

$$2\mathbf{\Omega} \times \mathbf{u}_{rot} = (0, 0, 2\Omega) \times (u_{rot}, v_{rot}, 0) = (-2\Omega v_{rot}, 2\Omega u_{rot}, 0)$$

the two horizontal components of Eq.(15) are:

$$\frac{du_{rot}}{dt} - 2\Omega v_{rot} = 0; \frac{dv_{rot}}{dt} + 2\Omega u_{rot} = 0 \tag{4}$$

$$u_{rot} = \frac{dx}{dt}; v_{rot} = \frac{dy}{dt}.$$

If we launch the puck from the origin of our coordinate system $x(0) = 0$; $y(0) = 0$ (chosen to be the center of the rotating dish) with speed $u_{rot}(0) = 0$; $v_{rot}(0) = v_o$, the solution to the above is:

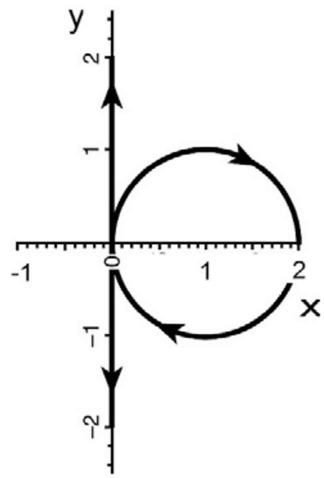

Figure 4: Trajectory of the puck studied in Section 3 in the inertial frame (straight line) and the rotating frame (circle). The scale of the axes are $\frac{v_o}{2\Omega}$. We launch the puck from the origin of our coordinate system $x(0) = 0$; $y(0) = 0$ (chosen to be the center of the rotating dish) with speed $u(0) = 0$; $v(0) = v_o$.

$$u_{rot}(t) = v_o \sin 2\Omega t;\ v_{rot}(t) = v_o \cos 2\Omega t$$

$$x(t) = \frac{v_o}{2\Omega} - \frac{v_o}{2\Omega} \cos 2\Omega t;\ y(t) = \frac{v_o}{2\Omega} \sin 2\Omega t$$

The puck's trajectory in the rotating frame is a circle - see Fig. 4 which should be compared with that observed in the experiment plotted in Fig.2b. The puck moves around a circle of radius of $\frac{v_o}{2\Omega}$ in a clockwise direction (anticyclonically) with a period $\frac{\pi}{\Omega}$.

3.2 Inertial frame

Now let us consider the same problem but in the non-rotating frame. The acceleration in a frame rotating at angular velocity $\mathbf{\Omega}$ is related to the acceleration in an inertial frame of reference by Eq.(9). And so, if the balance of forces is $\frac{d\mathbf{u}_{rot}}{dt} = -2\mathbf{\Omega} \times \mathbf{u}_{rot}$ these two terms cancel out in Eq.(9), and it reduces to:

$$\frac{d\mathbf{u}_{in}}{dt} = \mathbf{\Omega} \times \mathbf{\Omega} \times \mathbf{r}. \quad (5)$$

If the origin of our inertial coordinate system lies at the center of our dish, then the above can be written out in component form thus:

$$\frac{du_{in}}{dt} + \Omega^2 x = 0; \frac{dv_{in}}{dt} + \Omega^2 y = 0 \tag{6}$$

where $_{in}$ means inertial. This should be compared to the equation of motion in the rotating frame - see Eq.(4). Note that Eq.(6) is just Eq.(3).

The solution is:

$$u_{in}(t) = 0;\ v_{in}(t) = v_o \cos \Omega t$$

$$x_{in}(t) = 0;\ y_{in}(t) = \frac{v_o}{\Omega} \sin \Omega t$$

The trajectory in the inertial frame is a straight line - see Fig. 4. The length of the line is **twice** the diameter of the inertial circle and the frequency of the oscillation is **one-half** that observed in the rotating frame.

The above solutions go a long way to explaining what is observed in the experiments described above and expose many of the curiosities of rotating versus non-rotating frames of reference.

4 Appendix

4.1 Transformation in to rotating coordinates

Imagine that the puck in our rotating parabola experiment has velocity $\mathbf{u}_{in}$ in the inertial frame. Viewed on the rotating frame, however, it has velocity $\mathbf{u}_{rot}$. The two velocities are related through - as is evident from Fig. 5:

$$\mathbf{u}_{in} = \mathbf{u}_{rot} + \boldsymbol{\Omega} \times \mathbf{r} \ , \tag{7}$$

where $\mathbf{r}$ is the position vector of a parcel in the rotating frame and $\boldsymbol{\Omega} \times \mathbf{r}$ is the vector product of $\boldsymbol{\Omega}$ and $\mathbf{r}$. Here

$$\mathbf{u}_{in} = \left(\frac{d}{dt}\mathbf{r}\right)_{in} ; \quad \mathbf{u}_{rot} = \left(\frac{d}{dt}\mathbf{r}\right)_{rot}$$

where $\left(\frac{d}{dt}\mathbf{r}\right)_{in,\ rot}$ is the rate of change of position of the puck measured in the respective frames. Eq.(7) suggests the following 'rule' for transforming the rate of change of vectors between frames:

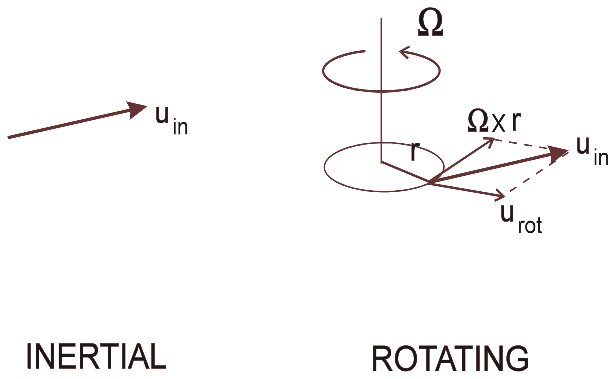

Figure 5: On the left is the velocity vector of a particle $\mathbf{u}_{in}$ in the inertial frame. On the right is the view from the rotating frame. The particle has velocity $\mathbf{u}_{rot}$ in the rotating frame. The relation between $\mathbf{u}_{in}$ and $\mathbf{u}_{rot}$ is $\mathbf{u}_{in} = \mathbf{u}_{rot} + \mathbf{\Omega} \times \mathbf{r}$ where $\mathbf{\Omega} \times \mathbf{r}$ is the velocity of a particle fixed (not moving) in the rotating frame at position vector $\mathbf{r}$.

$$\left(\frac{d}{dt}\right)_{in} = \left(\frac{d}{dt}\right)_{rot} + \mathbf{\Omega}\times \tag{8}$$

A more rigorous derivation of Eq.(8) can be found in Chapter 6 of the 12.003 notes.

Combining Eqs.(7) and (8) we see that:

$$\begin{aligned}\left(\frac{d\mathbf{u}_{in}}{dt}\right)_{in} &= \left(\frac{d}{dt}\right)_{in} (\mathbf{u}_{rot} + \mathbf{\Omega} \times \mathbf{r}) = \left(\left(\frac{d}{dt}\right)_{rot} + \mathbf{\Omega}\times\right)(\mathbf{u}_{rot} + \mathbf{\Omega} \times \mathbf{r}) \\ &= \left(\frac{d\mathbf{u}_{rot}}{dt}\right)_{rot} + 2\mathbf{\Omega} \times \mathbf{u}_{rot} + (\mathbf{\Omega} \times \mathbf{\Omega} \times \mathbf{r})\end{aligned} \tag{9}$$

Thus the equation of motion of the puck in the inertial frame is:

$$\left(\frac{d\mathbf{u}_{in}}{dt}\right)_{in} = \text{applied forces/unit mass} = \mathcal{F} \tag{10}$$

where, in the absence of friction,

$$\mathcal{F} = -g\widehat{\mathbf{z}} \tag{11}$$

is the gravitational acceleration acting on the puck, with $\widehat{\mathbf{z}}$ a unit vector in the vertical. Using Eq.(9), Eq.(10) can be written in the rotating frame thus:

$$\left(\frac{d\mathbf{u}_{rot}}{dt}\right)_{rot} = \underbrace{-2\boldsymbol{\Omega}\times\mathbf{u}}_{\substack{\text{Coriolis}\\ \text{accel}^n}} + \underbrace{-\boldsymbol{\Omega}\times\boldsymbol{\Omega}\times\mathbf{r}}_{\substack{\text{Centrifugal}\\ \text{accel}^n}} - g\widehat{\mathbf{z}} \tag{12}$$

Note that Eq.(12) is the same as Eq.(10) except that $\mathbf{u} = \mathbf{u}_{rot}$ and 'apparent' accelerations, introduced by the rotating reference frame, have been placed on the right-hand side of Eq.(12) [just as in the gradient wind equation that describes the radial inflow experiment, GFDIII]. The apparent accelerations are given names: the centrifugal acceleration ($-\boldsymbol{\Omega}\times\boldsymbol{\Omega}\times\mathbf{r}$) is directed radially outward - see Fig.5; the Coriolis acceleration ($-2\boldsymbol{\Omega}\times\mathbf{u}$) is directed 'to the right' of the velocity vector (if $\Omega > 0$ as sketched in Fig.6).

4.1.1 Centrifugal and Coriolis acceleration

Because centrifugal acceleration can be expressed as the gradient of a potential thus

$$-\boldsymbol{\Omega}\times\boldsymbol{\Omega}\times\mathbf{r} = \nabla\left(\frac{\Omega^2 r^2}{2}\right)$$

it is convenient to combine $\nabla\left(\frac{\Omega^2 r^2}{2}\right)$ with $g\widehat{\mathbf{z}} = \nabla(gz)$ - the gradient of the gravitational potential, gz - and write Eq.(12) in the succinct form:

$$\left(\frac{d\mathbf{u}_{rot}}{dt}\right)_{rot} = -2\boldsymbol{\Omega}\times\mathbf{u}_{rot} - \nabla\phi \tag{13}$$

where

$$\phi = gz - \frac{\Omega^2 r^2}{2} \tag{14}$$

is the modified (by centrifugal accelerations) gravitational potential 'measured' in the rotating frame.

Because our parabolic surface is constructed to ensure that ϕ=constant, $\nabla\phi = 0$, and so Eq.(13) reduces to:

$$\frac{d\mathbf{u}_{rot}}{dt} = -2\boldsymbol{\Omega}\times\mathbf{u}_{rot} \tag{15}$$

This is the equation of motion governing the puck on the parablolic surface in the rotating frame. With the signs shown, the parcel would turn to the right in response to the Coriolis force if $\Omega > 0$.

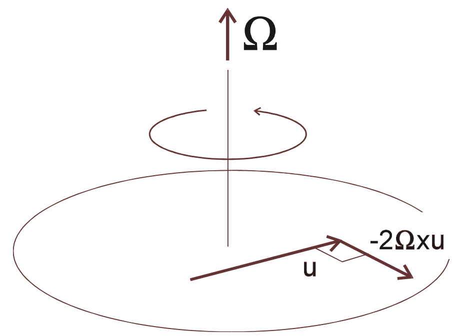

Figure 6: A fluid parcel moving with velocity u_{rot} in a rotating frame experiences a Coriolis acceleration $-2\Omega\times\ u_{rot}$, directed 'to the right' of u_{rot} if, as here, Ω is upwards, corresponding to anticyclonic rotation - like that of the northern hemisphere viewed from above the north pole; for the southern hemisphere, the sign of rotation is reversed and the deflection is to the left.

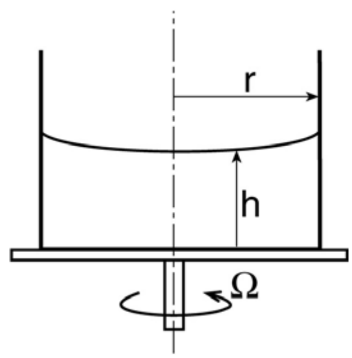

Figure 7: Water placed in a rotating tank and insulated from external forces (both mechanical and thermodynamic) eventually comes in to solid body rotation in which the fluid does not move relative to the tank. In such a state the free surface of the water is not flat but takes on the shape of a parabola given by Eq.(2).

4.2 The parabolic rotating table

Suppose we filled a tank with water, set it turning and leave it until it comes in to solid body rotation. We note that the free-surface of the water is not flat - it is depressed in the middle and rises up slightly to its highest point along the rim of the tank, as sketched in Fig. 7. What's going on?

In solid-body rotation, $\mathbf{u}_{rot} = \mathbf{0}$ and so Eq.(13) implies that $\nabla\phi = 0$ and so

$$gz - \frac{\Omega^2 r^2}{2} = \text{constant} \tag{16}$$

is just the modified gravitational potential, Eq.(14). We can determine the constant of proportionality by noting that at $r = 0$, $z = h(0)$, the height of the fluid in the middle of the tank. Hence the depth of the fluid h is given by Eq.(2). The free surface takes on a parabolic shape: it tilts so that it is always perpendicular to the vector $\mathbf{g}^*$ (gravity modified by centrifugal forces) given by $\mathbf{g}^* = -g\widehat{\mathbf{z}} - \boldsymbol{\Omega} \times \boldsymbol{\Omega} \times \mathbf{r}$. If we hung a plumb line in the frame of the rotating table it would point in the direction of $\mathbf{g}^*$ i.e. slightly outwards rather than directly down.

The parabola available in the lab was manufactured by pouring resin in to a mold on a table turning at rate $f = 2\Omega = 3\ rad\ s^{-1}$ and allowing it to set to form a highly smooth surface. Let us estimate the 'dip' of the free surface of the parabola by inserting numbers in to Eq.(**??**). If $f = 3$, as for the parabola available in the lab, $\Omega = 1.5\ s^{-1}$, the radius of the tank is 0.50 m, then with $g = 9.81$ m s^{-2}, we find $\frac{\Omega^2 r^2}{2g} \sim 2.9 \times 10^{-2}$m or about 3 cm, a noticeable effect.